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Renormalization of the noncommutative photon self-energy to all orders via Seiberg-Witten map

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Abstract: We show that the photon self-energy in quantum electrodynamics on noncommutative $\mathbb{R}^4$ is renormalizable to all orders (both in θ and $\hbar$) when using the Seiberg-Witten map. This is due to the enormous freedom in the Seiberg-Witten map which represents field redefinitions and generates all those gauge invariant terms in the θ-deformed classical action which are necessary to compensate the divergences coming from loop integrations.

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1. Introduction

Recently Noncommutative Yang-Mills (NCYM) theory has attracted considerable attention. Partly this is due to its role in string theory, where NCYM appears as a certain limit in presence of a constant background field B (see [1] and references therein). On the other hand, NCYM theory (or better: Yang-Mills theory on noncommutative $\mathbb{R}^4$) is also an example of gauge theory on a noncommutative algebra which is interesting on its own [2]. Actually the starting point was a combination of both [3].

Although renormalizable at the one-loop level [4, 5, 6], it became clear that noncommutative field theories suffer from a new type of infrared divergences [7, 8] which spoiled renormalization at higher loop order. Possible problems are ring-type divergences and commutants [9]. Although this analysis proved renormalizability for the Wess-Zumino model and complex scalar field theory [9], the situation for gauge theory was desperate.

An alternative approach to NCYM was proposed by Seiberg and Witten [1]. They argued from an equivalence of regularization schemes (point-splitting vs. Pauli-Villars) that there should exist a map (the so-called Seiberg-Witten map) which relates the noncommutative[1] gauge field $\hat{A}_\mu$ and the noncommutative gauge parameter $\hat{\lambda}$ to (local) counterparts A_ν and λ living on ordinary space-time. This approach was popularized in [10] where it was argued that this is the only way to obtain a finite number of degrees of freedom in non-abelian NCYM.

The Seiberg-Witten map leads to a gauge field theory with an infinite number of vertices and Feynman graphs with unbounded degree of divergence, which seemed to rule out a perturbative renormalization. An explicit quantum field theoretical investigation of the Seiberg-Witten map was first performed in [11] for noncommutative Maxwell theory. The outcome at one-loop for the photon self-energy was (to our surprise) gauge invariant and gauge independent. It was not renormalizable. However, the divergences were absorbable by gauge invariant extension terms to the classical action involving θ which we interpreted as coming from a more general scalar product.

It turns out that our extended action is actually a part of the Seiberg-Witten map when exploiting all its freedom, see also [12, 13]. This means that a renormalization of the Seiberg-Witten map itself is able to remove the one-loop divergences. This extends to a complete proof of all-order renormalizability of the photon self-energy. A generalization to other Green's functions is not obvious, however. This freedom in the Seiberg-Witten map can be regarded as a field redefinition.

2. The freedom in the Seiberg-Witten map

We consider NCYM theory with fermions, regarded as a model on ordinary Minkowski space (with metric $g^{\mu\nu}$), subject to the altered (non-local) multiplication law for functions f, g on space-time:

$$(f \star g)(x) = \int d^4y\, d^4z\; \delta(y-x)\,\delta(z-x)\;\exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\nu}\right)(f(y)g(z))\,. \qquad (2.1)$$

The real parameter $\theta_{\mu\nu} = -\theta_{\nu\mu}$ will be regarded as a constant external field of power-counting dimension -2.

The Seiberg-Witten map [1] expresses the noncommutative gauge fields $\hat{A}_\mu = \hat{A}_\mu[A_\nu, \theta]$, the infinitesimal gauge parameter $\hat{\lambda} = \hat{\lambda}[\lambda, A_\nu, \theta]$ and the fermions $\hat{\psi} = \hat{\psi}[\psi, A_\nu, \theta]$, $\hat{\bar{\psi}} = \hat{\bar{\psi}}[\bar{\psi}, A_\nu, \theta]$, which are multiplied according to (2.1), as formal power series of the corresponding gauge-equivalent commutative (but non-abelian)

[1] One should better say non-local instead of non-commutative because the $\star$-product is a non-local product between functions on space-time.

objects $A_\mu, \lambda, \psi, \bar\psi$ to be multiplied in the ordinary way. The gauge-equivalence condition is

$$\delta_{\hat\lambda} \hat A_\mu = \delta_\lambda \hat A_\mu\,, \qquad \delta_{\hat\lambda}\hat\psi = \delta_\lambda \hat\psi\,, \qquad \delta_{\hat\lambda}\hat{\bar\psi} = \delta_\lambda \hat{\bar\psi}\,, \tag{2.2}$$

with initial condition

$$\begin{aligned} \hat A_\mu[A_\nu, \theta=0] &= A_\mu\,, \qquad \hat\lambda[\lambda, A_\nu, \theta=0] = \lambda\,, \\ \hat\psi[\psi, A_\nu, \theta=0] &= \psi\,, \qquad \hat{\bar\psi}[\bar\psi, A_\nu, \theta=0] = \bar\psi\,. \end{aligned} \tag{2.3}$$

The noncommutative gauge transformations are defined by

$$\begin{aligned} \delta_{\hat\lambda}\Gamma = \int d^4x \Bigg(&\mathrm{tr}\Big((\partial_\mu \hat\lambda - \mathrm{i}(\hat A_\mu \star \hat\lambda - \hat\lambda \star \hat A_\mu)) \star \frac{\delta\Gamma}{\delta \hat A_\mu}\Big) + \\ &+ \Big\langle \frac{\Gamma\overleftarrow{\delta}}{\delta\hat\psi} \star (\mathrm{i}\hat\lambda \star \hat\psi)\Big\rangle + \Big\langle (-\mathrm{i}\hat{\bar\psi} \star \hat\lambda) \star \frac{\overrightarrow{\delta}\Gamma}{\delta\hat{\bar\psi}}\Big\rangle\Bigg) \end{aligned} \tag{2.4}$$

and the commutative[2] ones by

$$\delta_\lambda\Gamma = \int d^4\,x \left(\mathrm{tr}\left((\partial_\mu\lambda - \mathrm{i}(A_\mu\lambda - \lambda A_\mu))\frac{\delta\Gamma}{\delta A_\mu}\right) + \left\langle \frac{\Gamma\overleftarrow{\delta}}{\delta\psi}(\mathrm{i}\lambda\psi)\right\rangle + \left\langle (-\mathrm{i}\bar\psi\lambda)\frac{\overrightarrow{\delta}\Gamma}{\delta\bar\psi}\right\rangle\right). \tag{2.5}$$

The bracket $\langle\ \rangle$ means trace in colour and spinor space.

As shown in [12] there is a big variety of solutions of (2.2) and (2.3) corresponding to field redefinitions. Here we take a subclass of the solutions derived in [12].[3] We denote by $\hat A_\mu^{(n)}$ a solution of (2.2) and (2.3) up to order n in θ. Then, a further solution up to the same order n is obtained by adding any gauge-covariant term with *exactly*[4] n factors of θ,

$$\begin{aligned} \hat A_\mu^{(n)\prime} &= \hat A_\mu^{(n)} + \mathbb{A}_\mu^{(n)}\,, \\ \mathbb{A}_\mu^{(n)} &= \sum_{(i)} \kappa_i^{(n)} (\underbrace{g^{**}\cdots g^{**}}_{2n}\ \underbrace{\theta_{**}\cdots\theta_{**}}_{n}\ \underbrace{D_*\ldots D_*}_{l_1}(F_{**})\cdots \underbrace{D_*\ldots D_*}_{l_k}(F_{**}))_\mu^{(i)}\,, \end{aligned} \tag{2.6}$$

where $\sum_{j=1}^k l_j = 2n+1-2k$. This condition guarantees that $\hat A_\mu^{(n)\prime}$ has the same power-counting dimension[5] $(=1)$ as A_μ when taking θ of power-counting dimension -2. Each $*$ in (2.6) stands for a Lorentz index (all but the free lower index μ

[2] Although we are first of all interested in QED, we present everything as far as possible in a way which also applies to θ-deformed Yang-Mills theory.

[3] Similar ideas are used in [13] where a general formalism for the construction of the Seiberg-Witten map is given.

[4] This is important: $\mathbb{A}_\mu^{(n)}$ contains exactly n factors of θ whereas $\hat A_\mu^{(n)}$ contains $0 \le j \le n$ factors of θ.

[5] Power-counting dimensions dim are defined as follows: $\dim(A_\mu) = \dim(\hat A_\mu) = 1$, $\dim(\psi) = \dim(\hat\psi) = \dim(\bar\psi) = \dim(\hat{\bar\psi}) = 3/2$, $\dim(\partial_\mu) = 1$, $\dim(m) = 1$, $\dim\left(\int d^4x\right) = -4$, $\dim(\delta^4(x-y)) = 4$, $\dim\left(\int d^4p\right) = 4$, $\dim(\delta^4(p-q)) = -4$, $\dim(\theta) = -2$.

are summation indices). $D_\nu = \partial_\nu - \mathrm{i}[A_\nu, .\,]$ is the covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu]$ the (commutative) Yang-Mills field strength. The sum is over all index structures (i) and $\kappa_i^{(n)} \in \mathbb{R}$ is a free parameter. Inserted into the gauge-equivalence (2.2) there is on the l.h.s. at order n no further factor of θ coming from $\hat{\lambda}$ or the $\star$-product possible:

$$\delta_{\hat{\lambda}} \hat{A}_\mu^{(n)\prime} = \delta_{\hat{\lambda}} \hat{A}_\mu^{(n)} - \mathrm{i}[\mathbb{A}_\mu^{(n)}, \lambda] \equiv \delta_\lambda \hat{A}_\mu^{(n)\prime} \qquad \text{up to order } n\,. \tag{2.7}$$

Thus, $\hat{A}_\mu^{(n)\prime}$ is a solution of the gauge-equivalence condition if $\hat{A}_\mu^{(n)}$ is, up to order n. The effect of $\mathbb{A}_\mu^{(n)}$ on the noncommutative field strength $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - \mathrm{i}(\hat{A}_\mu \star \hat{A}_\nu - \hat{A}_\nu \star \hat{A}_\mu)$ is up to order n given by

$$\hat{F}_{\mu\nu}^{(n)\prime} = \hat{F}_{\mu\nu}^{(n)} + D_\mu \mathbb{A}_\nu^{(n)} - D_\nu \mathbb{A}_\mu^{(n)}\,, \tag{2.8}$$

because no factor θ from $\hat{A}_\alpha$ or the $\star$-product can be combined with $\mathbb{A}_\mu^{(n)}$ up to order n. The noncommutative Yang-Mills action is

$$\hat{\Sigma} = -\frac{1}{4g^2} \int d^4x \, \mathrm{tr}(\hat{F}^{\mu\nu} \hat{F}_{\mu\nu})\,. \tag{2.9}$$

Defining $\hat{\Sigma}^{(n)\prime}$ as the result of (2.9) when replacing $\hat{F}_{\mu\nu}^{(n)}$ by $\hat{F}_{\mu\nu}^{(n)\prime}$ and the commutative actions $\Sigma^{(n)\prime}$ and $\Sigma^{(n)}$ as the Seiberg-Witten map of $\hat{\Sigma}^{(n)\prime}$ and $\hat{\Sigma}^{(n)}$, we obtain up to order n in θ

$$\Sigma^{(n)\prime} = \Sigma^{(n)} + \frac{1}{g^2} \int d^4x \, \mathrm{tr}\left(F^{\mu\nu} D_\nu \mathbb{A}_\mu^{(n)}\right) = \Sigma^{(n)} + \frac{1}{g^2} \int d^4x \, \mathrm{tr}\left((D_\nu F^{\nu\mu}) \mathbb{A}_\mu^{(n)}\right). \tag{2.10}$$

The part $\Sigma^{(n)\prime} - \Sigma^{(n)}$ of the action represents due to (2.6) and the dimension assignment in footnote 5 a gauge invariant action of power-counting dimension 0 with n factors of θ. Gauge invariance means that application of the operator δ_λ defined in (2.5) yields zero. The action $\Sigma^{(n)}$ is gauge invariant at any order $k \leq n$ in θ, thus yielding at order n in θ terms which are also present in $\Sigma^{(n)\prime} - \Sigma^{(n)}$. These terms in $\Sigma^{(n)}$ can be regarded as a shift to $\kappa_i^{(n)}$.

Now we pass to quantum field theory and compute Feynman graphs. The loop integrations will produce divergent 1PI-Green's functions which under the assumption of an invariant renormalization scheme[6] are gauge invariant field polynomials of power-counting dimension 0. We hope to remove all of these divergences with n factors of θ by a $\hbar$-redefinition of $\kappa_i^{(n)}$. The problem is that (2.10) generates only a subset of all possible gauge invariant actions. For the photon self-energy in θ-deformed QED we are able to show that all divergences actually belong to this subset (section 5). Before we will address the question of a physical meaning of the $\kappa_i^{(n)}$.

[6]If no invariant renormalization scheme is available (or if one chooses a non-invariant scheme for some reason) one should attempt to restore gauge invariance via the quantum action principle and a parameter redefinition. Gauge anomalies are an obstruction to such a program.

3. Field redefinitions

It is possible to rewrite (2.10) in the following form:

$$\Sigma^{(n)\prime} = \Sigma^{(n)} + \delta^{(n)}_{\mathbb{A}}\Sigma^{(n)} \qquad \text{up to order } n\,,$$
$$\delta^{(n)}_{\mathbb{A}}\Gamma = \int d^4x\, \mathbb{A}^{a\,(n)}_\mu(x)\frac{\delta\Gamma}{\delta A^a_\mu(x)}\,, \tag{3.1}$$

where Γ is any functional depending on A^a_μ. In (3.1) we use now the component formulation induced by $A_\mu = A^a_\mu T_a$, with $[T_a, T_b] = \mathrm{i}f_{ab}{}^c T_c$. This suggests to consider $\mathbb{A}^{a\,(n)}_\mu$ as a field redefinition of A^a_μ. As such we must check how it commutes with the local Ward identity operator with respect to a variation of the gauge field

$$W^\lambda_a(y) = \frac{\delta}{\delta\lambda^a(y)}\delta_\lambda = -\partial^y_\mu\frac{\delta}{\delta A^a_\mu(y)} - f_{ab}{}^c A^b_\mu(y)\frac{\delta}{\delta A^c_\mu(y)}\,, \tag{3.2}$$

where δ_λ is defined in (2.5). To the commutator $\delta^{(n)}_{\mathbb{A}}W^\lambda_a(y)\Gamma - W^\lambda_a(y)\delta^{(n)}_{\mathbb{A}}\Gamma$ there is only a contribution if both operators $\delta^{(n)}_{\mathbb{A}}$ and $W^\lambda_a(y)$ hit the same field A_μ in Γ, hence it is sufficient to consider $\Gamma \mapsto A^c_\mu(x)$. Then we have

$$\delta^{(n)}_{\mathbb{A}}W^\lambda_a(y)A^c_\mu(x) = -f_{ab}{}^c\mathbb{A}^{b\,(n)}_\mu(y)\,\delta(x-y)$$

due to (3.2) and (3.1) and

$$W^\lambda_a(y)\delta^{(n)}_{\mathbb{A}}A^c_\mu(x) = W^\lambda_a(y)\mathbb{A}^{c\,(n)}_\mu(x) = -f_{ab}{}^c\mathbb{A}^{b\,(n)}_\mu(y)\,\delta(x-y)$$

because of the covariance $\delta_\lambda\mathbb{A}^{(n)}_\mu = \mathrm{i}[\lambda, \mathbb{A}^{(n)}_\mu]$, see (2.6). This means

$$[\delta^{(n)}_{\mathbb{A}}, W^\lambda_a(y)] \equiv 0\,, \tag{3.3}$$

i.e. all $\kappa^{(n)}_i$ in $\mathbb{A}^{(n)}_\mu$ must be regarded as parametrizations of field redefinitions.

4. Quantum field theory

The basic object in quantum field theory is the generating functional Γ of one-particle irreducible (1PI) Green's functions (with n factors of θ)

$$\Gamma[A_{c\ell}]^{(n)} = \sum_{N\geq 2}\frac{1}{N!}\int d^4x_1\cdots d^4x_N A^{a_1}_{\mu_1\,c\ell}(x_1)\cdots A^{a_N}_{\mu_N\,c\ell}(x_N)\times$$
$$\times\,\langle 0|TA^{\mu_1}_{a_1}(x_1)\ldots A^{\mu_N}_{a_N}(x_N)|0\rangle^{(n)}_{\mathrm{1PI}} \tag{4.1}$$

in terms of classical fields $A_{c\ell}$. Colour indices are denoted by a_i. The vacuum expectation value of the time-ordered product of fields in (4.1) is the Fourier transform of the N-point vertex functional in momentum space,

$$\langle 0|TA^{\mu_1}_{a_1}(x_1)\ldots A^{\mu_N}_{a_N}(x_N)|0\rangle^{(n)}_{\mathrm{1PI}} = \tag{4.2}$$
$$= \int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\delta^4(p_1+\cdots+p_N)\,\mathrm{e}^{-\mathrm{i}p_1x_1}\cdots\mathrm{e}^{-\mathrm{i}p_Nx_N}\,\Gamma^{\mu_1\ldots\mu_N\,(n)}_{a_1\ldots a_N}(p_1,\ldots,p_N)$$

with $\dim(\Gamma^{\mu_1 \cdots \mu_N}_{a_1 \ldots a_N}(p_1, \ldots, p_N)) = 4 - N$. Due to the n factors of θ, the momentum space degree of divergence of $\Gamma^{\mu_1 \cdots \mu_N \,(n)}_{a_1 \ldots a_N}(p_1, \ldots, p_N)$ is $\omega = 4 + 2n - N$. The local Ward identity operator (3.2) applied to (4.1),

$$W^\lambda_a(y)\Gamma[A_{c\ell}]^{(n)} = -\partial^y_\rho \frac{\delta \Gamma[A_{c\ell}]^{(n)}}{\delta A^a_{\rho\, c\ell}(y)} - f_{ab}{}^c \, A^b_{\rho\, c\ell}(y) \frac{\delta \Gamma[A_{c\ell}]^{(n)}}{\delta A^c_{\rho\, c\ell}(y)} \,, \tag{4.3}$$

is evaluated in presence of an invariant renormalization scheme to

$$W^\lambda_a(y)\Gamma[A_{c\ell}]^{(n)} = \partial^\mu \partial_\mu B_a(y) + f_{abc} A^b_{\mu\; c\ell}(y) \partial^\mu B^c(y) \,.$$

Here, B is the multiplier field required for gauge-fixing. In a linear gauge[7] there are no vertices with external B-lines and thus no divergent 1PI Green's functions with external B (furthermore, B is independent of θ). Therefore we have the local Ward identity $W^\lambda_a(y)\Gamma[A_{c\ell}]^{(n)} = 0$ for $\Gamma[A_{c\ell}]^{(n)}$ being 1PI and divergent. Then, functional derivation of (4.3) with respect to $A^{a_1}_{\mu_1\; c\ell}(x_1) \cdots A^{a_N}_{\mu_N\; c\ell}(x_N)$, followed by putting the remaining $A^b_{\nu\, c\ell}(z) = 0$, gives

$$\begin{aligned} 0 = & -\partial^y_\rho \langle 0 | T A^\rho_a(y) A^{\mu_1}_{a_1}(x_1) \ldots A^{\mu_N}_{a_N}(x_N) | 0 \rangle^{(n)}_{\text{1PI}} - \\ & - \sum_{j=1}^N f_{a a_j}{}^c \delta(y - x_j) \Big\langle 0 \Big| T A^{\mu_j}_c(y) A^{\mu_1}_{a_1}(x_1) \cdots \overset{j}{\check{}} \cdots A^{\mu_N}_{a_N}(x_N) \Big| 0 \Big\rangle^{(n)}_{\text{1PI}} . \end{aligned} \tag{4.4}$$

where $\overset{j}{\check{}}$ means omission of $A^{\mu_j}_{a_j}(x_j)$.

5. The photon self-energy in θ-deformed QED

We recall that $\omega = 4 + 2n - N$ is the power-counting degree of divergence for the N-point photon vertex functionals with n factors of θ, independent of the internal structure of the Feynman graphs. Due to translation invariance (or momentum conservation) we therefore have

$$\begin{aligned} & \langle 0 | T A^{\mu_1}(x_1) \ldots A^{\mu_N}(x_N) | 0 \rangle^{(n)}_{\text{1PI}} = \\ & = \sum_{(i)} \kappa'^{(n)}_i \Big(\underbrace{g^{**} \cdots g^{**}}_{2n+2} \; \underbrace{\theta_{**} \cdots \theta_{**}}_{n} \; \underbrace{\partial_* \cdots \partial_*}_{4+2n-N} (\delta(x_1 - x_2) \cdots \delta(x_{N-1} - x_N)) \Big)^{(i)\mu_1 \ldots \mu_N} . \end{aligned} \tag{5.1}$$

The sum is over all index structures (i) with appropriate numerical factors $\kappa'^{(n)}_i$, and the derivatives are with respect to any of the coordinates $x_1, \ldots, x_N$. We insert (5.1) into (4.1) and integrate by parts. Assuming an invariant renormalization scheme (such as dimensional regularization), the local Ward identity (4.4), with $f_{ab}{}^c = 0$,

[7] We refer to [11] for a natural nonlinear gauge in θ-deformed Maxwell theory.

implies that the generating functional $\Gamma[A_{c\ell}]^{(n)}$ must be a function of the classical field strength $F_{c\ell\,\mu\nu} = \partial_\mu A_{c\ell\,\nu} - \partial_\nu A_{c\ell\,\mu}$:

$$\Gamma[A_{c\ell}]^{(n)} = \sum_{N\geq 2} \frac{1}{N!} \int d^4x_1 \cdots d^4x_N \sum_{(i)} \kappa_i'^{(n)} \Big(\underbrace{g^{**}\cdots g^{**}}_{2n+2} \underbrace{\theta_{**}\cdots\theta_{**}}_{n} \times \\ \times \underbrace{\partial_* \ldots \partial_*}_{4+2n-2N} \left(F_{**c\ell}(x_1)\cdots F_{**c\ell}(x_N)\right) \delta(x_1-x_2)\cdots\delta(x_{N-1}-x_N)\Big)^{(i)} . \tag{5.2}$$

From the Ward identity it follows in particular that $N \leq n+2$.

Now we specialize (5.2) to the photon self-energy, i.e. to the $N=2$ part in (5.2). All derivatives can be assumed acting on $F_{**c\ell}(x_1)$. There are both $2n$ indices on θ_{**} and ∂_*, but we have $\theta^{\alpha\beta}\partial^{x_1}_\alpha \partial^{x_1}_\beta = 0$. Therefore, there is for $n \geq 1$ always one of the terms

$$\partial^\mu F_{\mu\nu} \qquad \text{or} \qquad \partial^\alpha\partial_\alpha F_{\mu\nu} = \partial_\mu \partial^\alpha F_{\alpha\nu} - \partial_\nu\partial^\alpha F_{\alpha\mu}$$

in the $N=2$ part of (5.2). But this is according to (2.10) nothing but the structure of a noncommutative Maxwell action after Seiberg-Witten map (with $D_\nu \equiv \partial_\nu$), which thus is able to absorb all divergences coming from loop integrations: the two-point function in the noncommutative Maxwell action

$$\hat\Sigma' = -\frac{1}{4g^2}\int d^4x\ \hat F^{\mu\nu\prime} \hat F'_{\mu\nu} \tag{5.3}$$

is renormalizable at order n in θ and any order L in $\hbar$ due to the gauge-covariant terms $\mathbb{A}^{(n)}_\mu$ in the Seiberg-Witten map, i.e. by a $\hbar$-redefinition of $\kappa_i^{(n)}$ which preserves the form of (5.3).

The argument does not work for N-point functions with $N \geq 3$. For instance, it is now possible to contract all derivatives in (5.2) with the factors of θ as the following contribution to the 3-point function shows:

$$\int d^4x_1\, d^4x_2\, d^4x_3 \theta^{\gamma\delta} \left(\prod_{i=1}^3 \theta^{\alpha_i\beta_i}\partial^{x_1}_{\alpha_i}\partial^{x_2}_{\beta_i}\right) \left(F_{\gamma\delta}(x_1) F^{\mu\nu}(x_2) F_{\mu\nu}(x_3)\right)\delta(x_1-x_2)\delta(x_2-x_3)\,.$$

The complete renormalization of NCYM theories remains an open problem.

5.1 One-loop photon self-energy at second order in θ

As an example let us look at the lowest orders of noncommutative Maxwell theory studied in [11]. In order θ^1 there is only one[8] gauge covariant (here: invariant) extension to the Seiberg-Witten map:

$$\mathbb{A}^{(1)}_\mu = \kappa_1^{(1)} \theta_{\mu\alpha}\partial_\beta F^{\alpha\beta}\,,$$

[8] The free index μ can not occur via ∂_μ because this would lead to a vanishing field strength. Moreover, one has to take the Bianchi identity into consideration.

which, however, drops out of the Maxwell action, $F^{\mu\nu}\theta_{\mu\alpha}\partial_\nu\partial_\beta F^{\alpha\beta} = -\theta_{\mu\alpha}(\partial_\nu F^{\mu\nu})\times(\partial_\beta F^{\alpha\beta}) = 0$. At order θ^2 we have, up to total derivatives $\partial_\mu(\,.\,)$ and Bianchi identity, four different terms[9] in (2.6):

$$\begin{aligned}\mathbb{A}^{(2)}_\mu &= \kappa^{(2)}_1 g^{\alpha\gamma}g^{\beta\delta}g^{\lambda\rho}g^{\sigma\tau}\theta_{\alpha\beta}\theta_{\gamma\delta}\partial_\lambda\partial_\rho\partial_\sigma F_{\tau\mu} + \kappa^{(2)}_2 g^{\alpha\gamma}g^{\beta\lambda}g^{\delta\rho}g^{\sigma\tau}\theta_{\alpha\beta}\theta_{\gamma\delta}\partial_\lambda\partial_\rho\partial_\sigma F_{\tau\mu} + \\ &\quad + \kappa^{(2)}_3 g^{\beta\sigma}g^{\gamma\tau}g^{\alpha\lambda}g^{\delta\rho}\theta_{\mu\beta}\theta_{\gamma\delta}\partial_\alpha\partial_\lambda\partial_\rho F_{\sigma\tau} + \kappa^{(2)}_4 g^{\gamma\tau}g^{\beta\delta}g^{\alpha\lambda}g^{\rho\sigma}\theta_{\mu\beta}\theta_{\gamma\delta}\partial_\alpha\partial_\lambda\partial_\rho F_{\sigma\tau}\,.\end{aligned} \tag{5.4}$$

These lead to the following terms in the action (2.10):

$$\begin{aligned}\Sigma^{(2)\prime} = \Sigma^{(2)} + \frac{1}{g^2}\int d^4x A_\mu\Big(&(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\Big(\kappa^{(2)}_1\theta^2\Box^2 + \kappa^{(2)}_2\tilde{\tilde{\Box}}\Box\Big) + \kappa^{(2)}_3\tilde{\partial}^\mu\tilde{\partial}^\nu\Box^2 + \\ &+ \kappa^{(2)}_4\Big(\theta^{\mu\alpha}\theta^\nu{}_\alpha\Box^3 + \Big(\tilde{\tilde{\partial}}^\mu\partial^\nu + \tilde{\tilde{\partial}}^\nu\partial^\mu\Big)\Box^2 + \partial^\mu\partial^\nu\tilde{\tilde{\Box}}\Box\Big)\Big)A_\nu\,,\end{aligned} \tag{5.5}$$

where $\Box = \partial^\alpha\partial_\alpha$, $\tilde{\partial}^\alpha = \theta^{\alpha\beta}\partial_\beta$, $\tilde{\tilde{\partial}}^\alpha = \theta^{\alpha\beta}\tilde{\partial}_\beta$, $\tilde{\tilde{\Box}} = \tilde{\partial}^\alpha\tilde{\partial}_\alpha$ and $\theta^2 = \theta^{\alpha\beta}\theta_{\alpha\beta}$. The r.h.s. of (5.5) can now be rewritten in the following form:

$$\frac{1}{g^2}\int d^4x\,\partial_\rho F^{\rho\mu}(x)\mathbb{A}^{(2)}_\mu(x) = \frac{1}{2g^2}\int d^4x\,d^4y\,A_\mu(x)A_\nu(y)\,\langle 0|T A^\mu(x)A^\nu(y)|0\rangle^{(2)}_{\rm 1PI}\,, \tag{5.6}$$

with

$$\begin{aligned}\langle 0|TA^\mu(x)A^\nu(y)|0\rangle^{(2)}_{\rm 1PI} = \Big(&(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\Big(2\kappa^{(2)}_1\theta^2\Box^2 + 2\kappa^{(2)}_2\tilde{\tilde{\Box}}\Box\Big) + 2\kappa^{(2)}_3\tilde{\partial}^\mu\tilde{\partial}^\nu\Box^2 + \\ &+ 2\kappa^{(2)}_4\Big(\theta^{\mu\alpha}\theta^\nu{}_\alpha\Box^3 + \Big(\tilde{\tilde{\partial}}^\mu\partial^\nu + \tilde{\tilde{\partial}}^\nu\partial^\mu\Big)\Box^2 + \partial^\mu\partial^\nu\tilde{\tilde{\Box}}\Box\Big)\Big)_x \times \\ &\times\delta(x-y)\,.\end{aligned} \tag{5.7}$$

Comparing (5.5) with the one-loop calculation in [11] we see that the following renormalization of $\kappa^{(2)}_1,\dots,\kappa^{(2)}_4$,

$$\begin{aligned}\kappa^{(2)}_1 &\mapsto \kappa^{(2)}_1 - \frac{g^2\hbar}{16(4\pi)^2\varepsilon}\,, & \kappa^{(2)}_2 &\mapsto \kappa^{(2)}_2 + \frac{7g^2\hbar}{40(4\pi)^2\varepsilon}\,, \\ \kappa^{(2)}_3 &\mapsto \kappa^{(2)}_3 + \frac{g^2\hbar}{60(4\pi)^2\varepsilon}\,, & \kappa^{(2)}_4 &\mapsto \kappa^{(2)}_4 + \frac{g^2\hbar}{8(4\pi)^2\varepsilon}\,,\end{aligned} \tag{5.8}$$

cancels precisely the one-loop divergences in the photon self-energy. In other words, (5.8) provides a formal power series $\kappa^{(2)}_i[\hbar]$ such that the one-loop photon self-energy Green's function is at order θ^2 renormalizable. However, (5.8) represent unphysical renormalizations because the κ's parametrize field redefinitions, see section 3. This means that at order 0 in $\hbar$ the $\kappa^{(2)}_i$ may be set to zero.

6. Extension to any order in θ

It remains to prove that the gauge-equivalence (2.2) of the Seiberg-Witten map can be extended to order $n+1$ in θ. This is not clear a priori because the gauge transformations $\delta_{\hat{\lambda}}$ and δ_λ applied to $\mathbb{A}^{(n)}_\mu$ produce very different results at higher order in θ.

[9] There are no divergent graphs of order 2 in θ with more than two external photon lines.

We expand $\hat{A}_\mu^{(n+1)}$ into a Taylor series:

$$\begin{aligned}\hat{A}_\mu^{(n+1)} &= \sum_{k=0}^{n+1} \frac{1}{k!}\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_k\beta_k}\left(\frac{\partial^k \hat{A}_\mu^{(n+1)}}{\partial\theta^{\alpha_1\beta_1}\ldots\partial\theta^{\alpha_k\beta_k}}\right)_{\theta=0} \\ &= A_\mu + \sum_{k=1}^{n+1}\frac{1}{k!}\theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\cdots\theta^{\alpha_k\beta_k}\left(\frac{\partial^{k-1}}{\partial\theta^{\alpha_2\beta_2}\ldots\partial\theta^{\alpha_k\beta_k}}\left(\frac{\partial\hat{A}_\mu^{(n+1)}}{\partial\theta^{\alpha_1\beta_1}}\right)\right)_{\theta=0} . \end{aligned} \tag{6.1}$$

We recall now the Seiberg-Witten differential equation[10] [1]

$$\frac{\partial\hat{A}_\mu}{\partial\theta^{\alpha\beta}} = -\frac{1}{8}\Big\{\hat{A}_\alpha,(\hat{F}_{\beta\mu}+\partial_\beta\hat{A}_\mu)\Big\}_\star + \frac{1}{8}\Big\{\hat{A}_\beta,(\hat{F}_{\alpha\mu}+\partial_\alpha\hat{A}_\mu)\Big\}_\star \tag{6.2}$$

for a solution $\hat{A}_\mu$ of (2.2), where $\{X,Y\}_\star := X\star Y + Y\star X$ is the $\star$-anticommutator. We see that $\partial\hat{A}_\mu^{(n+1)}/\partial\theta^{\alpha\beta}$ requires knowledge of only $\hat{A}_\nu^{(n)}$ (i.e. of the Seiberg-Witten map up to order n). Taking the general order-n solution (2.6), i.e. including $\mathbb{A}_\nu^{(n)}$, we obtain a Seiberg-Witten map up to order $n+1$,

$$\begin{aligned}\hat{A}_\mu^{(n+1)\prime} = A_\mu - \frac{1}{4}\sum_{k=1}^{n+1}\frac{1}{k!}\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_k\beta_k}\left(\frac{\partial^{k-1}}{\partial\theta^{\alpha_2\beta_2}\ldots\partial\theta^{\alpha_k\beta_k}}\Big\{\hat{A}_{\alpha_1}^{(n)\prime},\Big(\hat{F}_{\beta_1\mu}^{(n)\prime}+\partial_{\beta_1}\hat{A}_\mu^{(n)\prime}\Big)\Big\}_\star\right)_{\theta=0} + \\ + \mathbb{A}_\mu^{(n+1)} ,\end{aligned} \tag{6.3}$$

which implies renormalizability up to order $n+1$ in θ. Accordingly, the noncommutative gauge parameter is at order $n+1$ in θ obtained as

$$\hat{\lambda}^{(n+1)} = \lambda - \frac{1}{4}\sum_{k=1}^{n+1}\frac{1}{k!}\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_k\beta_k}\left(\frac{\partial^{k-1}}{\partial\theta^{\alpha_2\beta_2}\ldots\partial\theta^{\alpha_k\beta_k}}\Big\{\hat{A}_{\alpha_1}^{(n)\prime},\partial_{\beta_1}\hat{\lambda}^{(n)}\Big\}_\star\right)_{\theta=0} .$$

Thus we have proved by induction that the photon self-energy arising from the noncommutative Maxwell action (5.3) is (under the assumption of an invariant renormalization scheme) renormalizable to all orders in θ and $\hbar$ via a general Seiberg-Witten map. Observe that $\hat{A}_\mu^{(n+1)\prime}$ is a complicated nonlinear function of $\kappa_i^{(j)}$ for $j \leq n$.

7. Remarks on the fermionic action

We would like to extend the renormalizability proof for the photon self-energy to Green's functions in θ-deformed QED [14] containing fermions. So far we did not succeed, nevertheless we present some ideas which hopefully turn out to be useful. On that level we can formulate everything for Yang-Mills theory with fermions.

[10] We would like to stress that (6.2) guarantees $\dim(\hat{A}_\mu) = 1$ to all orders of θ.

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In analogy to (2.6) we add to a solution $\hat{\psi}^{(n)}$ of the gauge-equivalence (2.2) the most general gauge-covariant term in ψ with exactly n factors of θ:

$$
\begin{aligned}
\hat{\psi}^{(n)\prime} &= \hat{\psi}^{(n)} + \Psi^{(n)}\,, \\
\Psi^{(n)} &= \sum_{(i)} \tilde{\kappa}_i^{(n)} \Big(m^t \underbrace{\theta_{**} \cdots \theta_{**}}_{n} \langle \bar{\psi} P^{r^1}_{l_0^1 l_1^1 \dots l_{k_1}^1} \psi\rangle \cdots \langle \bar{\psi} P^{r^s}_{l_0^s l_1^s \dots l_{k_s}^s} \psi\rangle\, P^{r^0}_{l_0^0 l_1^0 \dots l_{k_0}^0} \psi \Big)^{(i)}\,, \\
P^{r^j}_{l_0^j l_1^j \dots l_{k_j}^j} &= \underbrace{\gamma^* \cdots \gamma^*}_{r^j} \underbrace{D_* \dots D_*}_{l_1^j} (F_{**}) \cdots \underbrace{D_* \dots D_*}_{l_{k_j}^j} (F_{**})\, \underbrace{\tilde{D}_* \cdots \tilde{D}_*}_{l_0^j}\,,
\end{aligned}
\tag{7.1}
$$

where $\sum_{j=0}^s (2k_j + \sum_{h=0}^{k_j} l_h^j) = 2n - t - 3s$ and $\sum_{j=0}^s r^j = 4n - t - 3s$. These conditions guarantee that $\hat{\psi}^{(n)\prime}$ has the same power-counting dimension ($= 3/2$) as ψ. All indices are summation indices. We have introduced the covariant derivative for fermions $\tilde{D}_\mu \psi = \partial_\mu \psi - \mathrm{i} A_\mu \psi$, m is the fermion mass and γ^μ are the Dirac gamma matrices. The quantity $\langle \bar{\psi} P^r_{l_0 l_1 \dots l_k} \psi \rangle$ is a (gauge invariant) function on space-time obtained by taking the trace in spinor and colour space, without space-time integration.

In the same way as in (2.7), $\hat{\psi}^{(n)\prime}$ is a solution of the gauge-equivalence (2.2) if $\hat{\psi}^{(n)}$ is:

$$
\delta_{\hat{\lambda}} \hat{\psi}^{(n)\prime} = \delta_{\hat{\lambda}} \hat{\psi}^{(n)} + \mathrm{i}\lambda \Psi^{(n)} = \delta_\lambda \hat{\psi}^{(n)\prime} \qquad \text{up to order } n. \tag{7.2}
$$

The Seiberg-Witten map for the adjoint spinor $\bar{\psi} = \psi^\dagger \gamma^0$ is simply obtained by hermitean conjugation, using $\gamma^{\mu\dagger}\gamma^0 = \gamma^0 \gamma^\mu$: a term

$$
\tilde{\kappa}\; \Pi \gamma^{\mu_1} \cdots \gamma^{\mu_r} P^0_{0 l_1^0 \dots l_{k_0}^0} \tilde{D}_{\nu_1} \dots \tilde{D}_{\nu_l} \psi
$$

in $\Psi^{(n)}$, where Π contains all saturated fermions $\langle \bar{\psi} P^r_{l_0 l_1 \dots l_k} \psi \rangle$, is transformed into

$$
\bar{\tilde{\kappa}}\; (\tilde{D}^\dagger_{\nu_1} \dots \tilde{D}^\dagger_{\nu_l})(\bar{\psi})\, \gamma^{\mu_r} \cdots \gamma^{\mu_1} P^0_{0 l_{k_0}^0 \dots l_1^0} \bar{\Pi}
$$

in $\bar{\Psi}^{(n)}$, where $\tilde{D}^\dagger_\nu \bar{\psi} = \partial_\nu \bar{\psi} + \mathrm{i}\bar{\psi} A_\nu$.

Then, the noncommutative Dirac action

$$
\hat{\Sigma}_D = \int d^4x \, \Big(\langle \hat{\bar{\psi}} (\mathrm{i}\gamma^\mu \partial_\mu - m) \hat{\psi} \rangle + \langle \hat{\bar{\psi}} \gamma^\mu \hat{A}_\mu \star \hat{\psi} \rangle \Big) \tag{7.3}
$$

gives after Seiberg-Witten map the real-valued gauge invariant fermionic action

$$
\Sigma_D^{(n)\prime} = \Sigma_D^{(n)} + \int d^4x \Big(\langle \bar{\psi} (\gamma^\mu (\mathrm{i}\partial_\mu + A_\mu) - m) \Psi^{(n)} \rangle + \langle \bar{\Psi}^{(n)} (\gamma^\mu (\mathrm{i}\partial_\mu + A_\mu) - m) \psi \rangle \Big)\,. \tag{7.4}
$$

The part $\Sigma_D^{(n)\prime} - \Sigma_D^{(n)}$ is due to (7.1) a real-valued gauge invariant integrated field polynomial of power-counting dimension 0 with at least two fermions. Such terms will also come from the action $\Sigma_D^{(n)}$, which leads effectively to a shift of $\tilde{\kappa}_i^{(n)}$. However, this generates only a subset of all gauge invariant fermionic actions [15]. The hope is that (assuming again an invariant renormalization scheme) the (divergent) 1PI Green's functions are precisely of the form (7.4). As for the N-point photon functions with $N \geq 3$, the Ward identity gives no further information.

Assuming it is possible to prove that divergent 1PI Green's functions are of the form (7.4), let us show that the Seiberg-Witten map (7.1) for fermions can be extended to order $n+1$. This goes as in the bosonic case via Taylor expansion and the differential equation implementing the gauge-equivalence:

$$\frac{\partial\hat{\psi}}{\partial\theta^{\alpha\beta}} = -\frac{1}{8}\Big(2\hat{A}_\alpha \star \partial_\beta\hat{\psi} - \partial_\alpha\hat{A}_\beta \star \hat{\psi}\Big) + \frac{1}{8}\Big(2\hat{A}_\beta \star \partial_\alpha\hat{\psi} - \partial_\beta\hat{A}_\alpha \star \hat{\psi}\Big)\,. \tag{7.5}$$

Then,

$$\begin{aligned}\hat{\psi}^{(n+1)\prime} = \psi - \frac{1}{4}\sum_{k=1}^{n+1}\frac{1}{k!}\theta^{\alpha_1\beta_1}\cdots\theta^{\alpha_k\beta_k} \times \\ \times \left(\frac{\partial^{k-1}}{\partial\theta^{\alpha_2\beta_2}\ldots\partial\theta^{\alpha_k\beta_k}}\Big(2\hat{A}^{(n)\prime}_{\alpha_1}\star\partial_{\beta_1}\hat{\psi}^{(n)\prime} - \partial_{\alpha_1}\hat{A}^{(n)\prime}_{\beta_1}\star\hat{\psi}^{(n)\prime}\Big)\right)_{\theta=0} + \\ + \Psi^{(n+1)}\end{aligned} \tag{7.6}$$

is the required solution of the gauge-equivalence at order $n+1$ in θ. Again, $\hat{\psi}^{(n+1)\prime}$ is a complicated nonlinear function of $\kappa_i^{(j)}$ and $\tilde{\kappa}_i^{(j)}$ for $j \leq n$.

8. Discussion

We have proved renormalizability of the photon self-energy in noncommutative QED to all orders in perturbation theory. This is the first example of a renormalizable Green's function in a noncommutative gauge theory. After the classification of diseases of noncommutative QFTs by Chepelev and Roiban [9] there remained not much hope that this could be achieved beyond one-loop.

The alternative approach via the Seiberg-Witten map [1] introduces an infinite number of non-renormalizable vertices with unbounded power-counting degree of divergence into the game. It is therefore surprising that at least for the photon self-energy such bad divergences can be treated. Fortunately the Seiberg-Witten map is a friendly monster which for each problem in a given order provides a cure in the same order (by shifting the mess to the next order, etc).

In this way we have achieved renormalization of a Green's function in a gauge theory with an external field of negative power-counting dimension — a model with infinitely many vertices. The point is that via the Seiberg-Witten map all these vertices can be summed up to an action as simple as (5.3). There exist closed formulae for the Seiberg-Witten map to all orders in θ, see [16] and references therein. In [16] there was also given an abstract definition of the freedom in the Seiberg-Witten map which should contain the field redefinitions we used to show renormalizability of the photon self-energy. It should be stressed however that only concrete loop calculations such as done in [11] can determine the parametrization (5.8) which renormalizes the photon self-energy.

Of course the renormalizability proof should be extended to other Green's functions than the photon self-energy. This is an open problem, but it is plausible now that noncommutative QED is renormalizable. Indeed, the photon self-energy contains (at high enough loop order) graphs of any other Green's function as subdivergences. These subdivergences assumed to be treated according to the forest formula, we know that the overall divergence of the photon self-energy is renormalizable. The open question is whether the Green's functions of these subdivergences can give rise to counterterms incompatible with the noncommutative action after field reparametrizations.

A main goal is of course to formulate a renormalizable noncommutative version of the standard model. In this respect we stress that in θ-deformed QED there is only one place for a coupling constant — namely in front of the photon action. It is therefore not possible to have fermions of different electric charge [17]. This is not a problem because in noncommutative geometry a part of the electric charge of the quarks comes from the colour sector [18].

One of the basic principles of renormalization is the independence of the specific way one treats the problems. How can we understand then the UV/IR problem [7, 8] which plagues the θ-undeformed approach and which is completely absent in the Seiberg-Witten framework? We believe that the UV/IR mixing is not really there, it is a non-perturbative artefact absent in perturbation theory — and thus should be treated by non-perturbative techniques as suggested in [9]. Let us consider the integral

$$I = \int d^4k \, \frac{e^{i\tilde{p}_\mu k^\mu}}{k^2} \,, \qquad \tilde{p}_\mu := \theta_{\mu\nu} p^\nu \,,$$

which is part of the tadpole graph in noncommutative Maxwell theory. The standard integration methods agree in the following (finite!) answer:

$$I = \int d^4k \, \frac{e^{i\tilde{p}_\mu k^\mu}}{k^2} = \frac{4\pi^2}{\tilde{p}^\mu \tilde{p}_\mu} \,. \tag{8.1}$$

This $(1/p^2)$ behaviour is the origin of all infrared problems. On the other hand, expanding the exponential we produce at first sight divergences of arbitrary degree:

$$I = \int d^4k \, \frac{\sum_{n=0}^\infty \frac{1}{n!}(i\tilde{p}_\mu k^\mu)^n}{k^2} \,.$$

Exchanging the sum and the integration, the integral of any term in the series is scale-independent and IR well-behaved — and as such zero in all standard renormalization schemes:

$$I = \sum_{n=0}^\infty \frac{1}{n!} \int d^4k \, \frac{(i\tilde{p}_\mu k^\mu)^n}{k^2} = 0 \,. \tag{8.2}$$

The infrared problem disappeared. There is no contradiction between (8.1) and (8.2) because the integral is clearly not absolutely convergent so that exchanging sum and integration is dangerous. Which one of (8.1) and (8.2) is correct? There are good reasons to believe that the θ-perturbative result (8.2) should be preferred — it leads to a renormalizable photon self-energy. In some sense this can be regarded as a normal ordering in noncommutative renormalization: first the integrals must be performed, then the sums. This eliminates the infrared singularities.

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